9th lecture + 10th lecture

Curl of a vector field:-

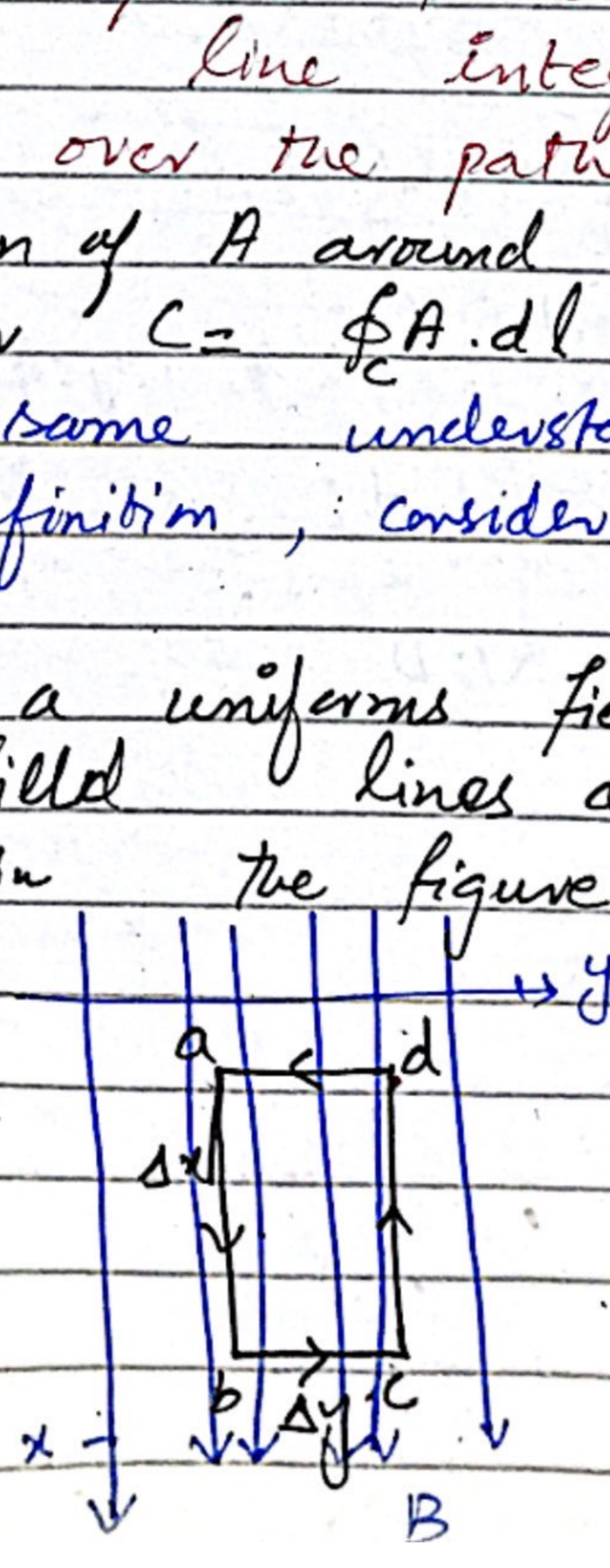
We have stated that a net outward flux of a vector A through a surface bounding a volume indicates the presence of a source. This source may be called a flow source, and div A is a measure of the strength of the flow source. There is another kind of source called vortex source, which causes a circulation of a vector field around it.

The net circulation or simply circulation of a vector field around a closed path is defined as the scalar line integral of the vector over the path. We have

Circulation of A around contour $C = \oint_C A \cdot dl$

To gain some understanding of this definition, consider the following examples.

1) Consider a uniform field $B = B_0 \hat{a_x}$, whose field lines are as depicted in the figure given below.



For the rectangular contour abcd shown in the figure, we have

$$\text{circulation} = \int_a^b (B_0 \hat{a}_x)(dx\,\hat{a}_x) + \int_b^c (B_0 \hat{a}_x)(dy\,\hat{a}_y) + \int_c^d -(B_0 \hat{a}_x)(dx\,\hat{a}_x) + \int_d^a (B_0 \hat{a}_x).(dy\,\hat{a}_y)$$

$$= \int_a^b (B_0 \hat{a}_x)(dx\,\hat{a}_x) + \int_c^d -(B_0 \hat{a}_x)(dx\,\hat{a}_x)$$

$$= B_0 x \Big|_a^b - B_0 x \Big|_c^d$$

$$= B_0(b-a) - B_0(d-c)$$

$\therefore$ $b-a = c-d = \Delta x$ so,

$$= 0$$

and because $\hat{a}_x . \hat{a}_y = 0$ so the second and fourth integrals are zero. So, we conclude that the circulation of uniform field is zero.

2 Example:-

Now, consider the magnetic field B induced by an infinite wire carrying a d-c current I. If the current is in the free space and its is oriented along the z-direction, then the magnetic flux density is given as

$$B = \frac{\mu_0 I}{2\pi r} \hat{a}_\varphi$$

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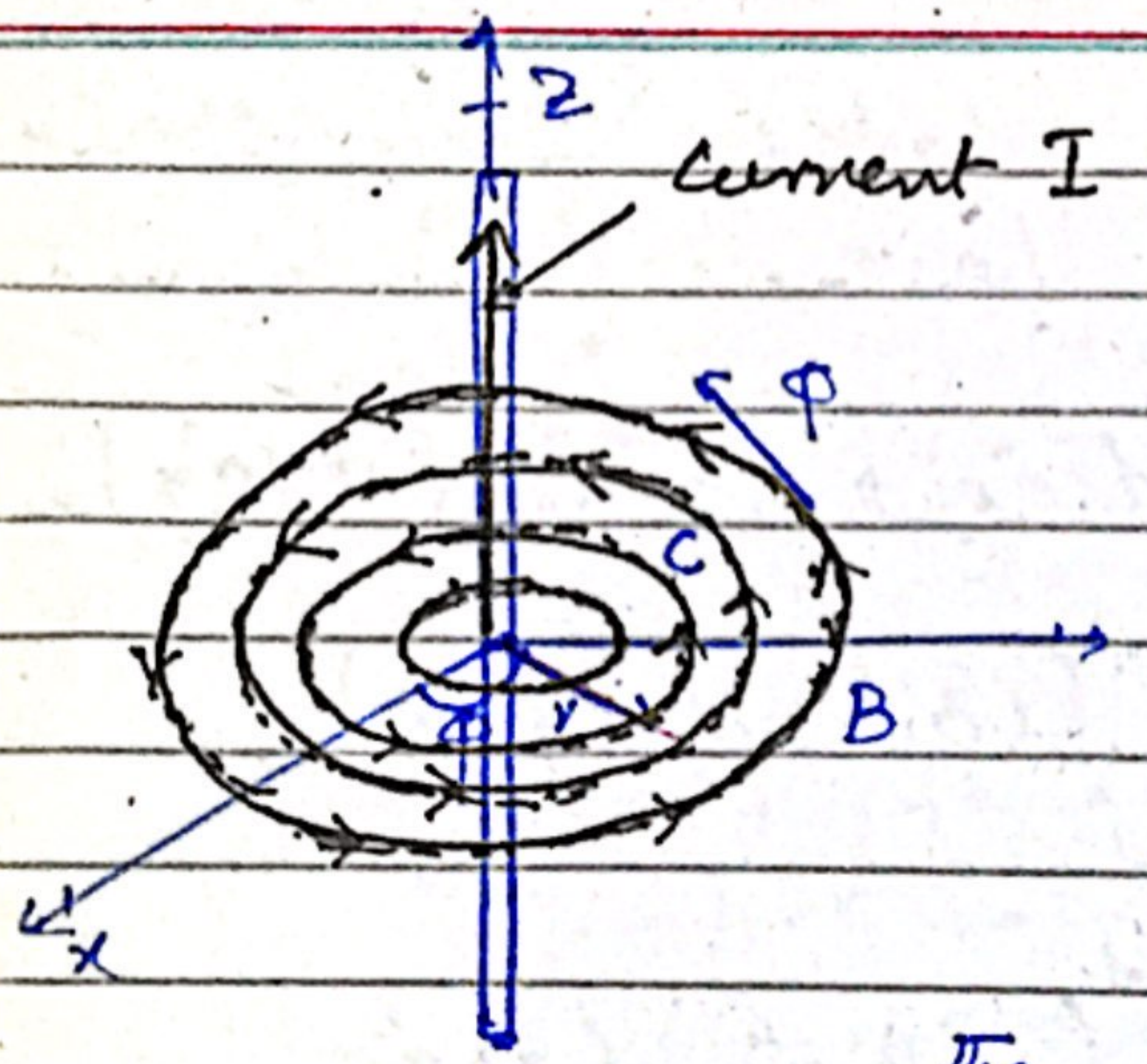


where μ_0 is the permeability of free space. and r is the radial distance from the current in the xy plane. The direction of B is along the increasing angle of φ.

The field lines of B are concentric circles around the current source as shown in above figure. For a circular contour of radius r, the differential length vector $dl = r\,d\varphi\,\hat{a}_\varphi$ and the circulation of B around C is

$$\text{circulation} = \oint_C B\cdot dl.$$

$$= \int_0^{2\pi} \left(\frac{\mu_0 I}{2\pi r}\hat{a_\varphi}\right)\left(r\,d\varphi\,\hat{a}_\varphi\right)$$

$$= \int_0^{2\pi} \frac{\mu_0 I}{2\pi}\, d\hat{\varphi}$$

$$= \frac{\mu_0 I}{2\pi}\cdot \varphi\Big|_0^{2\pi}$$

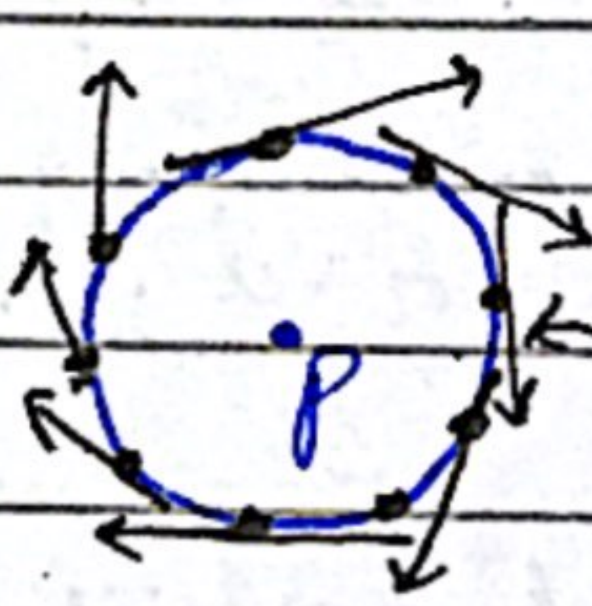
$$= \frac{\mu_0 I}{2\pi}\cdot[2\pi]$$

$$= \mu_0 I$$ (Here the circulation is not zero)

The magnitude of the circulation of B depends on the choice of contour. Also, the direction of the contour determines whether the circulation is +ve or -ve.

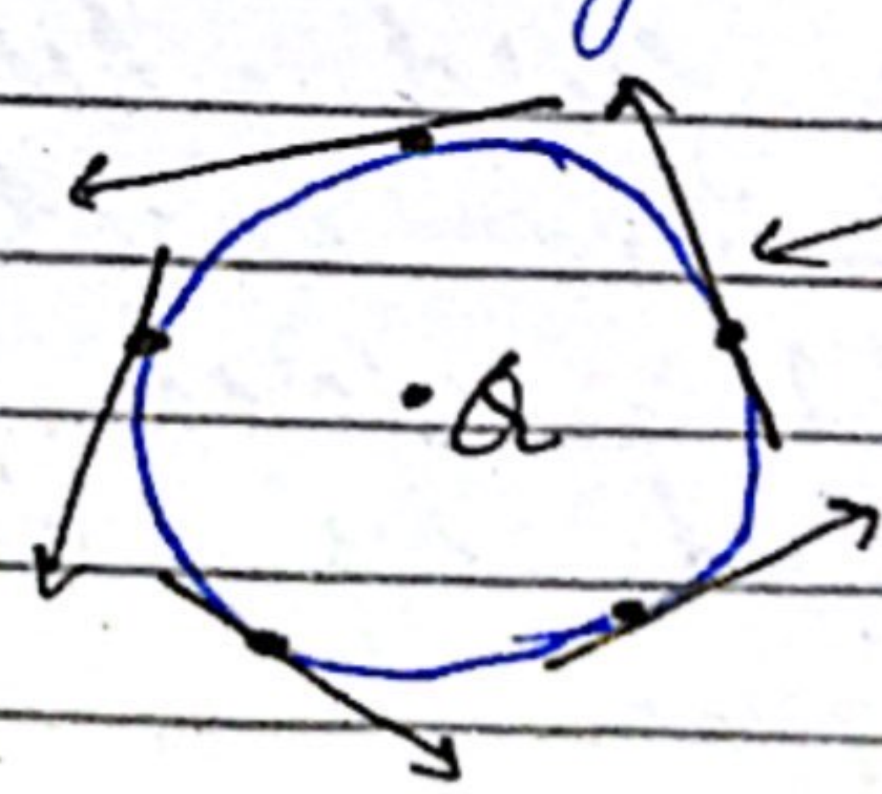
Explanation:-

If you consider a point in a vector field and make an elementary circle centered around that point then assuming equally spaced selective points over the periphery of the elementary circle if the associated vectors are tangential and develop a clockwise rotational tendency, then point P is said to possess +ve curl as shown in the figure below.



Associated vector
(+ve curl)

Similarly, if the associated vectors at selective points are tangential and develop an anticlockwise rotational tendency, the point Q is said to possess a negative curl as shown below.



Associated vectors
(-ve curl)

Generally, the curl of a vector field $\vec{A}$ at a point is a vector pointing in the direction of a normal to an infinitesimal surface which is so oriented in space that the limit of ratio of the line integral of the vector field $\vec{A}$ around the perimeter of that ~~clos~~ surface to the area enclosed is maximal. The magnetude of the curl is the value of that limit.

Mathematically, the curl vector $\vec{C}$ of the vector field $\vec{A}$ is by this definition determined by

$$\vec{C} = \text{curl } \vec{A} = \lim_{\Delta s \to 0} \frac{\oint \vec{A} \cdot d\vec{l}}{\Delta s} \Bigg|_{max} \cdot \hat{u}_n$$

where $\hat{u}_n$ is the normal to that surface at point P in the right hand sense with ~~re~~ respect to the direction of the closed-line integral.

In order to calculate the curl of a vector field, lets choose a rectangular coordinates, and an incremental closed path of sides Δx and Δy as shown in next figure.

As we cause the closed path to shrink, the preceding expression becomes more nearly exact, and in the limit we have the equality

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\oint H \cdot dL}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}$$

Now, if we choose closed paths which are oriented perpendicularly to each of the remaining two coordinates axes, analogous processes lead to expressions for the x and y components of

$$\lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}} \frac{\oint H \cdot dL}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \quad \text{— (ii)}$$

and

$$\lim_{\Delta z, \Delta x \to 0} \frac{\oint H \cdot dL}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \quad \text{— (iii)}$$

so, we can write

$$\operatorname{curl}(H)_N = \lim_{\Delta s \to 0} \frac{\oint H \cdot dL}{\Delta s}$$

In rectangular coordinates the x, y and z components of the curl H are given by (i), (ii) and (iii) and therefore

$$\text{Curl}(H) = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{a}_z$$

The result may also be written in the form of a determinant

$$\text{Curl}(H) = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

or

$$\text{Curl } H = \nabla \times H$$

The expression for the curl of the vector field $\vec{H}$ in the cylindrical and spherical coordinate systems, respectively are

$$\nabla \times \vec{H} = \frac{1}{\rho} \begin{vmatrix} \vec{a_\rho} & \rho \vec{a_\phi} & \vec{a_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix}$$

and

$$\nabla \times \vec{H} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{a_r} & r\vec{a_r} & r\sin\theta \vec{a_\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & r\sin\theta H_\phi \end{vmatrix}$$

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The physical significance of the curl of a vector field is that it represents the circulation per unit area of the vector field taken around a small area of any shape. Its direction is normal to the plane of the surface. Stated differently, if the line integral of a vector field about a closed elementary path is nonzero, the curl of a vector field is also non zero and we say that the vector field is rotational. The flow of water out of a tub or a sink provides an excellent example of a rotational velocity field of the flow. On the other hand, if the curl of a vector field is zero, the vector field is said to be irrotational and conservative

2.21 Q If $f(x,y,z)$ is a continuously differentiable scalar function. Show that $\nabla \times (\nabla f) = 0$.

Stoke's theorem:-

The stoke's Theorem relates the line integral to a surface integral. It states that " The line integral of $\vec{F}$ around a closed path L is equal to the integral of curl of $\vec{F}$ over the open surface S enclosed by the closed path L.

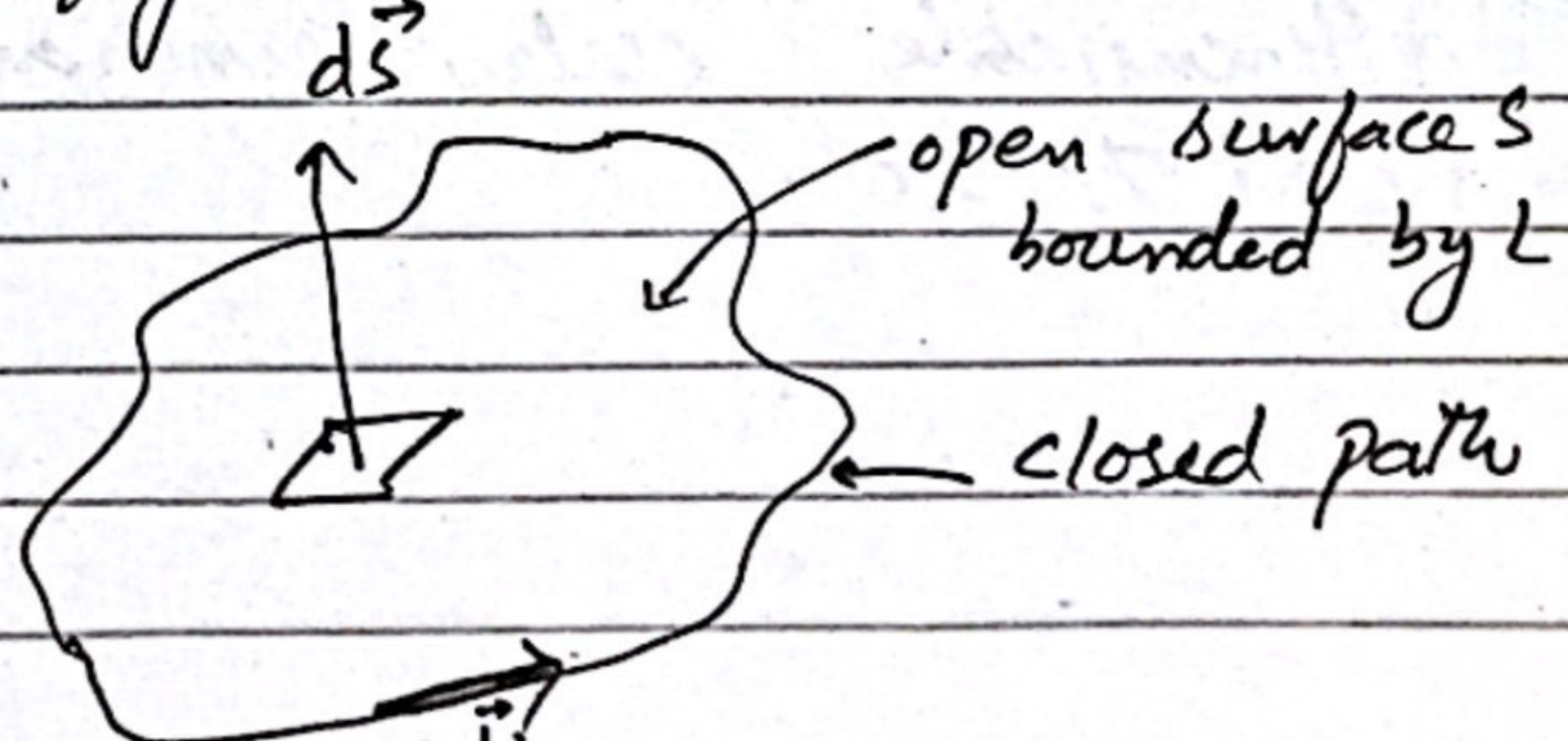
Mathematically it is expressed as,

$$\oint_L \vec{F} \cdot dL = \oint (\nabla \times \vec{F}) \cdot d\vec{s}$$

$d\vec{L}$ = Perimeter of total surface.

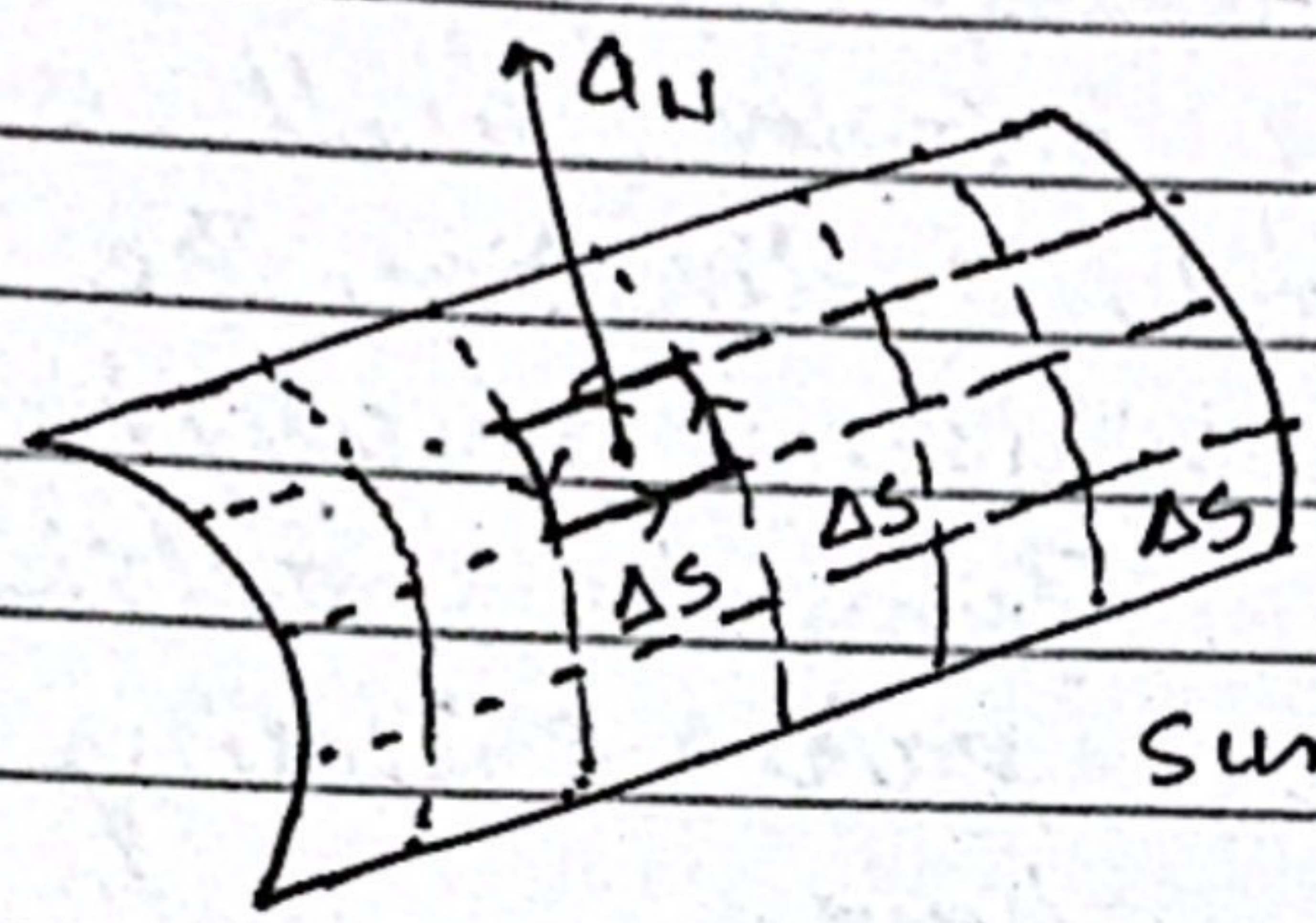
Key point:

Stoke's Theorem is applicable only when $\vec{F}$ and $\nabla \times \vec{F}$ are continuous on the surface S. The path L and open surface S enclosed by path L for which stoke's theorem is applicable is shown in the figure below.



Proof:-

Consider the surface S of the following figure which is broken up into incremental surfaces of area ΔS



If we apply the definition of the curl to one of these incremental surfaces, then

$$\oint \frac{H \cdot dL_{\Delta S}}{\Delta S} = (\nabla \times H)_N$$

where N is subscript indicates the right hand normal to the surface. The subscript ΔS indicates that the closed path is the perimeter of an incremental area ΔS. The result may also be written as

$$\oint \frac{H \cdot dL_{\Delta S}}{\Delta S} = (\nabla \times H) \cdot a_N$$

$$\int H \cdot dL_{\Delta S} = (\nabla \times H) \cdot a_N \Delta S$$
$$= (\nabla \times H) \Delta S$$

where a_N is a unit vector in the direction of the right hand normal to ΔS.

Now lets determine this circulation for every ΔS comprising S and sum the results. As we evaluate the closed line integral for each ΔS, some cancellation will so occur because every interior wall is covered once in each direction. The only boundaries on which cancellation can not occur from the outside boundary, the path enclosing S. Therefore, we have

$$\oint H.dL = \oint_S (\nabla \times H).dS$$

where dL is taken only on the perimeter of S.

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Practice Questions:-

Q:- For the scalar function $V = xy - z^2$, determine its directional derivative along the direction of vector $A = (a\hat{x} - z\hat{a_y})$ and then evaluate it at $P(1, -1, 2)$.

Q For the scalar function $T = e^{-r/5} \cos\phi$, determine its directional derivative along the radial direction $\hat{a_r}$ and then evaluate it at $P(2, \pi/4, 3)$

Q For the vector field $E = xz\hat{a_x} - yz^2\hat{a_y} - xy\hat{a_z}$, verify the divergence theorem by computing

(a) The total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the cartesian axes.

Ans: $\oint E.ds = -8/3$

(b) The integral $\nabla . E$ over the cube's volume. (Ans $\iiint \nabla . E dv = -8/3$

Q A vector field $D = r^3\hat{a_r}$ exists in the region b/w two concentric cylindrical surfaces bounded by $r=1$ and $r=2$; with both cylinders extending b/w $z=0$ and $z=5$. Verify the divergence theorem.

Ans = 150π

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Q) Given $V = x^2y + xy^2 + xz^2$ (a) find the gradient of V and (b) Evaluate it at $(1, -1, 3)$

Ans: $\nabla V = (2xy + y^2 + z^2)\hat{a}_x + (x^2 + 2xy)\hat{a}_y + 2xz\hat{a}_z$; $\nabla V|_{(1,-1,3)} = 8\hat{a}_x - \hat{a}_y + 6\hat{a}_z$.

Q) Find the directional derivative of $V = rz^2\cos 2\phi$ along the direction $A = 2\hat{a}_r - \hat{a}_z$ and evaluate it at $(1, \frac{\pi}{2}, 2)$

Ans: $\left(\frac{dv}{dl}\Big| = \frac{-4}{\sqrt{5}}\right)$

Q Given $A = e^{-2y}(\sin 2x\,\hat{a}_x + \cos 2x\,\hat{a}_y)$ Find $\nabla \cdot A$.

Ans : $\nabla \cdot A = 0$

Q) Given $A = r\cos\phi\,\hat{a}_r + r\sin\phi\,\hat{a}_\phi + 3z\hat{a}_z$ find $\nabla \cdot A$ at $(2, 0, 3)$

Ans = $\nabla \cdot A = 6$.

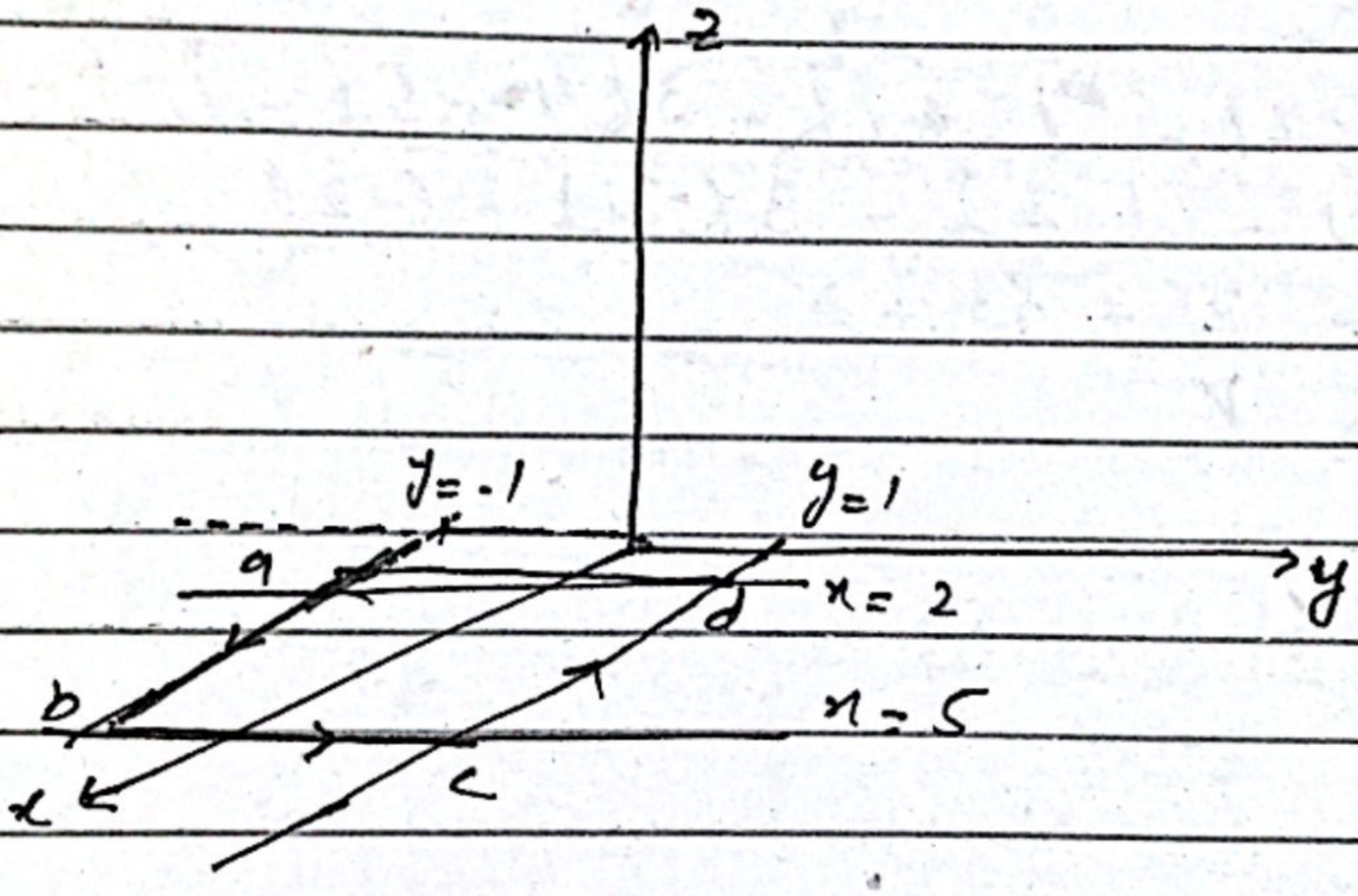
Q Find the gradient of r where r is the magnitude of the position vector $\vec{r} = \rho\vec{a}_\rho + z\vec{a}_z$ in the cylinderical coordinate system.

Evaluate both sides of the Stoke's theorem for the field $\vec{H} = 6xy\hat{a}_x - 3y^2\hat{a}_y$ A/m and the rectangular path around the region $2 \le x \le 5$; $-1 \le y \le 1$; $z = 0$. Let the +ve direction of $d\vec{S}$ be a_z

Sol

According to stoke's theorem

$$\oint \vec{H}\cdot d\vec{L} = \int_S (\nabla \times \vec{H})\, d\vec{S}$$



$$\oint \vec{H}\cdot d\vec{L} = \int_{ab} + \int_{bc} + \int_{cd} + \int_{da}$$

$$= \int_2^5 (6xy\hat{a}_x)(dx\hat{a}_x) + \int_{-1}^{1} (-3y^2\hat{a}_y)(dy\hat{a}_y)$$

$$+ \int_5^2 (-6xy\hat{a}_x)(dx\hat{a}_x) + \int_1^{-1} (+3y^2\hat{a}_y)(dy\hat{a}_y)$$

$$= \int_2^5 \cancel{-6x\,dx} + \int_{-1}^{1} 3y^2\,dy + \int_2^5 \cancel{6x\,dx} + \int_{-1}^{1} 3y^2\,dy$$

$$= 2\int 3y^2\,dy$$

Q Given $\vec{A} = \frac{10r^3}{4}\hat{a_r}$ in the cylinderical coordinate, evaluate both sides of the Divergence Theorem for the volume enclosed by $r=1$ and $r=2\,m$, $z=0$ and $10m$

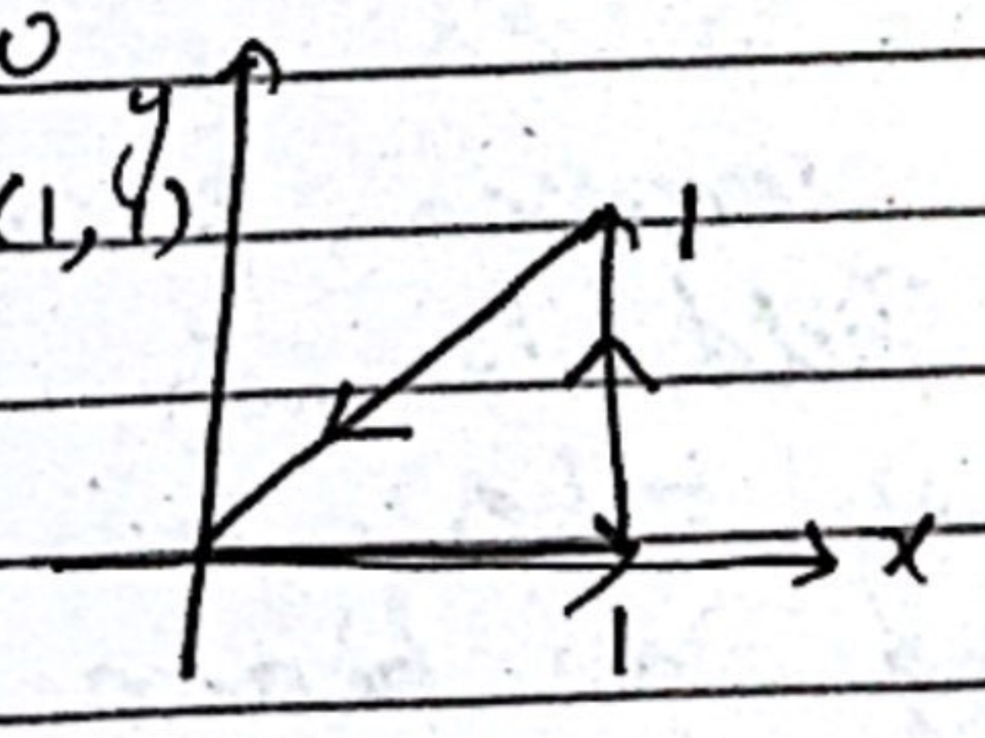
Ans: (750π)

Q Given that $A = 30e^{-r}\hat{a_r} - 2z\hat{a_z}$, in cylinderical coordinates, evaluate both sides of the Divergence Theorem for the volume enclosed by $r=2$, $z=0$ and $z=5$.

Ans: 129.4

Q For the vector Field $E = xy\hat{a_x} - (x^2+2y^2)\hat{a_y}$ calculate the following

(a) $\oint_c E.dl$ around the triangular contour shown in the figure below



(b) $\int_s (\nabla \times E).ds$ over the area of the triangle.

Ans: -1

Q- If $\vec{F} = (2z+5)\hat{a_x} + (3x-2)\hat{a_y} + (4x-1)\hat{a_z}$, verify Stokes theorem over the hemisphere $x^2+y^2+z^2 = 4$ and $z \geq 0$

Ans - $\oint_c \vec{E}.\vec{dl} = \int (\nabla \times \vec{F}) \cdot \vec{ds} = 12\pi$

Q. Vector $\vec{P}$ is given in different coordinates as below. Find the nature of field by obtaining divergence and curl in each case as given below.

(a) $\vec{P} = 30\hat{a_x} + 2xy^2\hat{a_y} + 5x^2z^2\hat{a_z}$

(b) $\vec{P} = \left(\frac{150}{r^2}\right)\hat{a_r} + 5\hat{a_\phi}$

Q. Find a unit vector which is normal to the surface given by $xy^2+3xz^2=4$ at a point $(1,-3,2)$

(Hint: Gradient of function gives a vector that is normal to a surface)

Q. Find constants a, b and c for a vector $\vec{P}\ (2x+y+az)\hat{a_x} + (bx+4y-3z)\hat{a_y} + (2x+cy+3z)\hat{a_z}$ to be irrotational

(Hint: For irrotational field $\nabla \times \vec{P} = 0$)

Q. For $\vec{F} = 2x^3\hat{a_x} + 2y^3\hat{a_y} + 2z^3\hat{a_z}$, evaluate $\oint \vec{F}.\vec{ds}$ using Gauss Divergence Theorem. Here S is the surface of sphere given as

$$x^2+y^2+z^2 = r^2$$

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⇒ The directional derivative of a function at a point in any direction is equal to the dot product of the gradient of the function and the unit vector in that direction.

Expression for the gradient of a scalar function in the cylinderical and spherical coordinate systems.

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{a}_\phi + \frac{\partial f}{\partial z}\hat{a}_z$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{a}_\phi$$

7th Lecture:-

Divergence of a vector Field

The spatial derivatives lead to the definition of the gradient. Now lets turn our attention to the spatial derivatives of a vector field. This results the definition of the divergence and the curl of a vector.

In the study of vector fields it is convenient to represent field variations graphically by directed field lines, which are called the flux lines or stream lines. They are directed lines or curves that indicate

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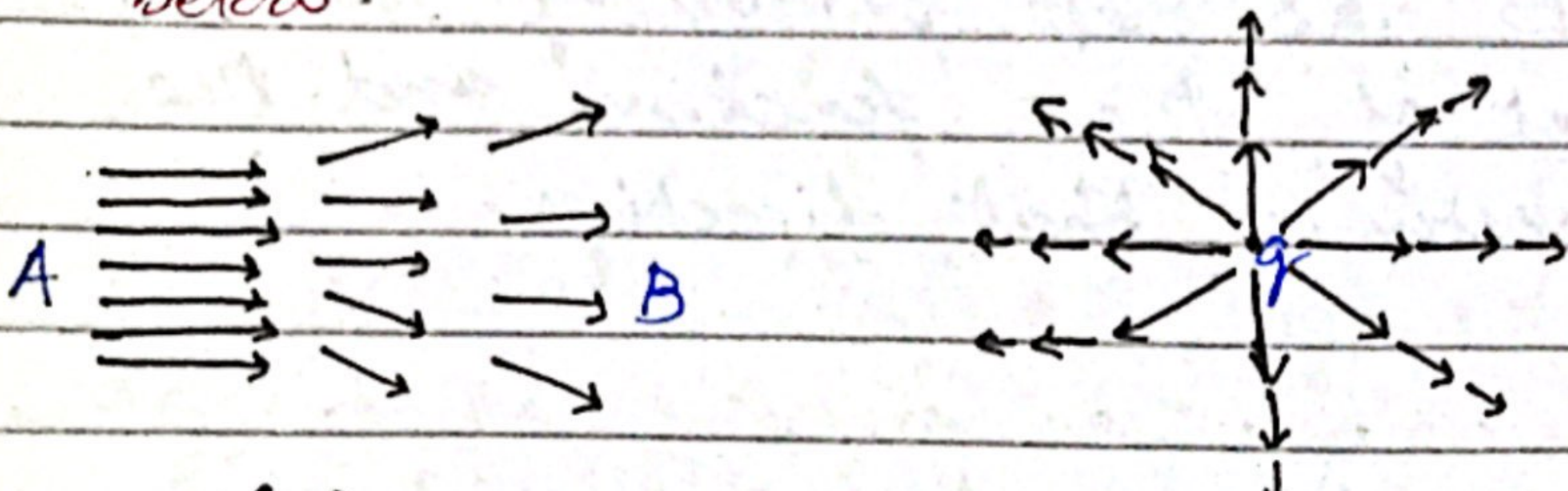
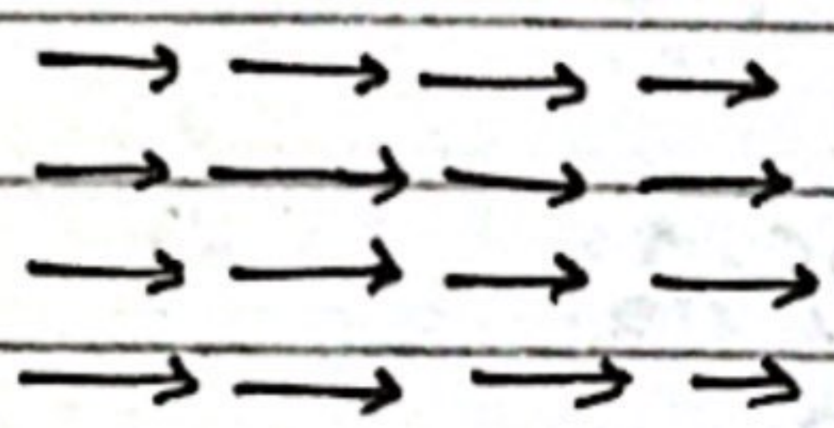
at each point the direction of the vector field, as shown in fig below.



(a) (b)


(c)

The magnitude of the field at a point is depicted either by the density or by the length of the directed lines in the vicinity of the point.

Fig (a) shows that the field in region A is stronger than that in region B. because there is a higher (arrow ~~(lengths away from A)~~ density of equal length directed lines in region A.

In fig (b), the decreasing arrow lengths away from point q indicate a radial field that is strongest in region closest to q.

fig (c) depicts a uniform field.

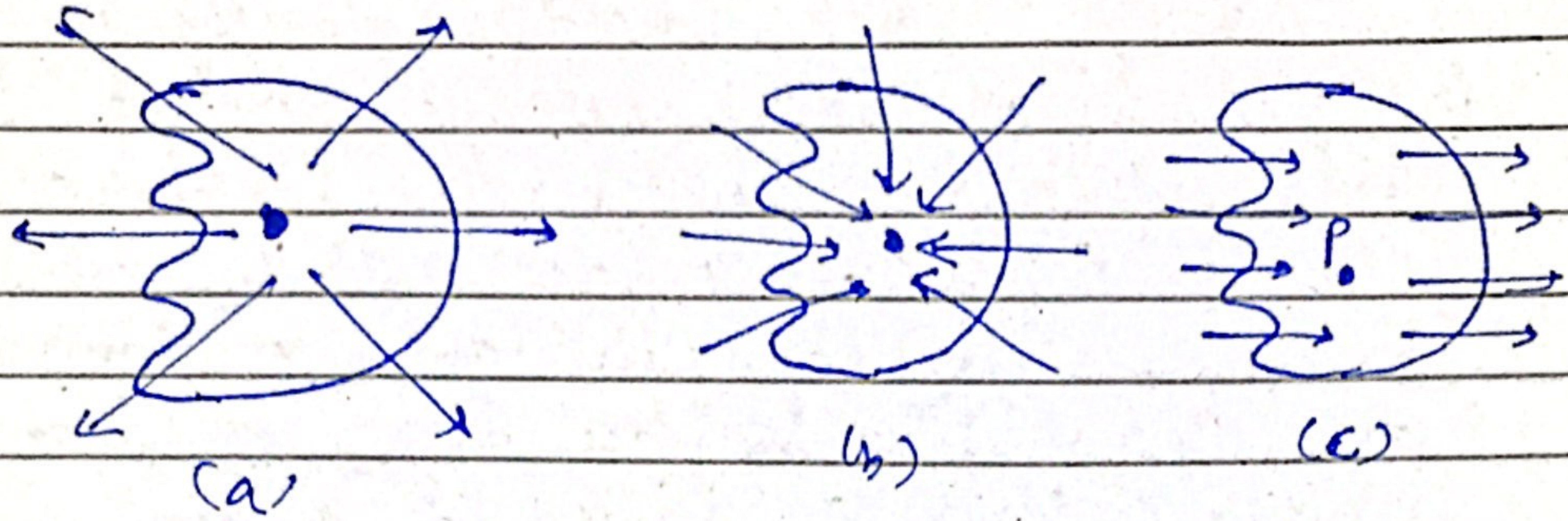
The vector field strength in fig (a) is measured by the number of flux lines passing through a unit ~~vector~~ surface normal to the vector.

The Flux of a vector field is analogous to the Flow of an incompressible fluid such as water. For a volume with an enclosed surface there will be an excess of outward or inward Flow through the surface only when a volume contains a source or a sink respectively. That is, a net +ve divergence indicates the presence of a source of fluid inside the volume, and a net -ve divergence indicates the presence of a sink.

~~eg~~ e.g., in fig(c) there is an equal amount of outward and inward flux going through any closed volume containing no sources of sinks, resulting in a zero divergence.

The divergence of a vector at a given point in a vector field is a scalar and is defined as the amount of Flux diverging from a unit vector element per second around that point.

The divergence of a vector at a point may be +ve if field lines are diverging or coming out from a small volume surrounding the point, as shown in next figure.



On the other hand, if field lines are converging into a small volume surrounding the point, the divergence of a vector is negative. [Fig b].
If the rate at which field lines are entering into a small volume surrounding the point is equal to the rate at which these are leaving that small volume, then the divergence of a vector is zero.

Derivation:-

Before defining the divergence of a vector field let us specify a scalar field f at point P in terms of a vector field $\vec{F}$ as

$$f = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int \vec{F} \cdot \vec{ds} \quad (i)$$

where the point P is enclosed by volume δV bounded by surface s.

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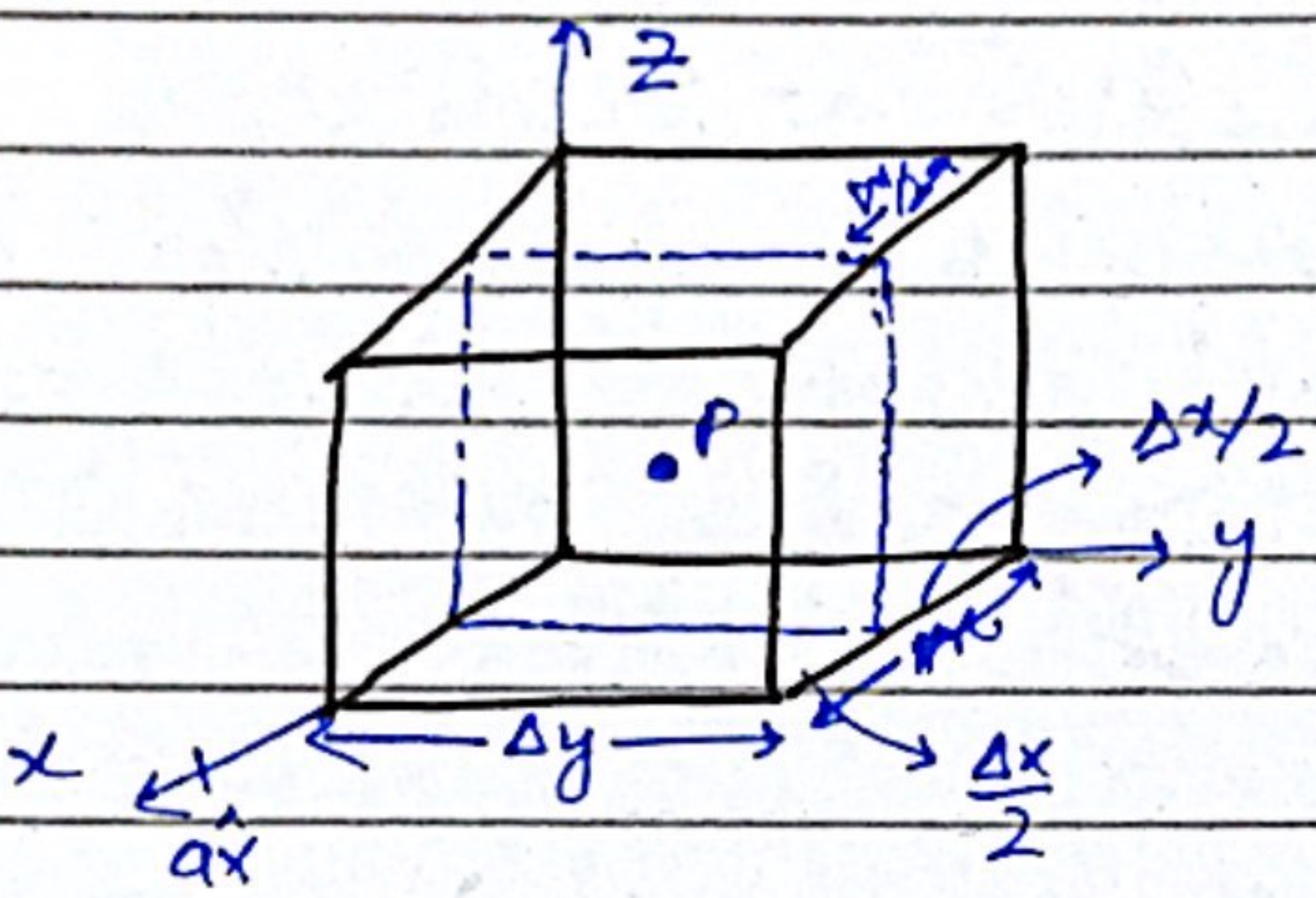
Although Δv can be of any shape, we construct a parallelepiped with sides Δx, Δy and Δz as shown in the figure below.



Note that $\vec{F}\cdot\vec{ds}$ defines the outward flow of the vector field $\vec{F}$ through the surface $\vec{ds}$ as the unit normal to ds points away from the volume enclosed. Thus $\oint \vec{F}\cdot\vec{ds}$ gives the net outward flow of flux of a vector field $\vec{F}$ from the volume Δv. However, the outward flow of a vector field $\vec{F}$ through the face in the +ve x-direction ~~using~~ is given as

$$\left[F_x + \frac{\partial F_x}{\partial x}\cdot\frac{\Delta x}{2}\right]\Delta y \Delta z$$

The outward flow of the vector field $\vec{F}$ through the surface in the negative x direction is

$$\left(-F_x + \frac{\partial F_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z$$

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and

$$\nabla\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 F_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta F_\theta] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}F_\phi$$

Question:-

Prove that $\nabla\cdot\vec{r} = 3$ where r is the position vector of any point in space.

Sol The position vector of any point P in rectangular coordinates is

$$\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

then

$$\nabla\cdot\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1+1+1$$

$$= 3$$

Gauss's Divergence Theorem:- (8th lecture)

The divergence of a vector field is defined as the net outward flux per unit volume. We may expect intuitively that the

"volume integral of the divergence of a vector field equals the total flux of the vector through the surface that bound the volume, that is,

$$\int_v \nabla\cdot A\, dv = \oint_s A\cdot ds$$

91

by. [illegible] is given

$$\frac{dT}{dl} = \nabla T . \hat{a}_l$$

if ∇T is a known function of the coordinate variables of a given coordinate system, we can find the difference $(T_2 - T_1)$, where T_1 and T_2 are the values of T at points P_1 and P_2 respectively, by integrating both sides of eq (iii). Thus

$$T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \, dl$$

Example :-

Find the directional derivative of $T = x^2 + y^2 z$ along the direction $2\hat{a}_x + 3\hat{a}_y - 2\hat{a}_z$ and evaluate it at $(1, -1, 3)$

Sol

The directional derivative is given as

$$\frac{dT}{dl} = \nabla T \hat{a}_l$$

First find the gradient of T

$$\nabla T = \left(\frac{\partial}{\partial x} \hat{a_x} + \frac{\partial}{\partial y} \hat{a_y} + \frac{\partial}{\partial z} \hat{a_z} \right)(x^2 + y^2 z)$$

$$= 2x \hat{a}_x + 2yz \hat{a}_y + y^2 \hat{a_z}$$

Denote $\underline{I}$ as the given direction,

$$\underline{I} = 2\hat{a}_x + 3\hat{a_y} - 2\hat{a_z}$$

Its unit vector is

$$\hat{a_l} = \frac{I}{|I|} = \frac{2\hat{a_x} + 3\hat{a_y} - 2\hat{a_z}}{\sqrt{17}}$$

$$\frac{dT}{dl} = \nabla T \cdot \hat{a_l} = \left(2x\hat{a_x} + 2yz\hat{a_y} + y^2\hat{a_z}\right)\left(\frac{2\hat{a_x} + 3\hat{a_y} - 2\hat{a_z}}{\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}}\left[4x + 6yz - 2y^2\right]$$

At $(1,-1,3)$

$$\left.\frac{dT}{dl}\right|_{(1,-1,3)} = \frac{1}{\sqrt{17}}\left[(4)(1) + 6(-1)(3) - 2(-1)^2\right]$$

$$= \frac{1}{\sqrt{17}}\left[4 - 18 - 2\right]$$

$$= \frac{-16}{\sqrt{17}}$$

Summary of properties of the gradient of a scalar function at a point:-

⇒ It is normal to the surface on which the given function is constant.

⇒ It points in the direction in which the given function changes more rapidly with position.

⇒ Its magnitude gives the max rate of change of the given function per unit distance.

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